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Hill's Equation with Random Forcing Parameters: Determination of Growth Rates Through Random Matrices

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Abstract This paper derives expressions for the growth rates for the random 2×2 matrices that result from solutions to the random Hill's equation. The parameters that appear in Hill's equation include the forcing strength q_k and oscillation frequency λ_k. The development of the solutions to this periodic differential equation can be described by a discrete map, where the matrix elements are given by the principal solutions for each cycle. Variations in the (q_k, λ_k) lead to matrix elements that vary from cycle to cycle. This paper presents an analysis of the growth rates including cases where all of the cycles are highly unstable, where some cycles are near the stability border, and where the map would be stable in the absence of fluctuations. For all of these regimes, we provide expressions for the growth rates of the matrices that describe the solutions.

Keywords Hill's equation · Random matrices · Lyapunov exponents

1 Introduction

This paper considers the growth rates for Hill's equation with parameters that vary from cycle to cycle. In this context, Hill's equation takes the form

$$\frac{d^2y}{dt^2} + [\lambda_k + q_k \hat{Q}(t)]y = 0, \tag{1}$$

where the barrier shape function $\hat{Q}(t)$ is periodic, so that $\hat{Q}(t + \Delta\tau) = \hat{Q}(t)$, where $\Delta\tau$ is the period. Here we take $\Delta\tau = \pi$, and the function $\hat{Q}$ is normalized so that $\int_0^{\Delta\tau} \hat{Q} dt = 1$. The forcing strength parameters q_k are a set of independent identically distributed (i.i.d.)

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random variables that take on a new value every cycle (where the index k labels the cycle). The parameters λ_k, which determine the oscillation frequency in the absence of forcing, also vary from cycle to cycle (and are i.i.d.). In principal, the cycle interval $\Delta\tau$ could also vary; however, this generalized case can be reduced to the problem of (1) through an appropriate re-scaling of the other parameters (see Theorem 1 of [1]).

Hill's equations [9] with constant values of the parameters have been well studied and arise in a wide variety of applications [12]. The introduction of parameters that sample a distribution of values is thus a natural generalization of this classic problem. Here we refer to the case with constant parameters as the "classical regime" of the general case.

For this class of periodic differential equations, the transformation that maps the coefficients of the principal solutions from one cycle to the next takes the form

$$\mathcal{M}_k = \begin{bmatrix} h_k & (h_k^2-1)/g_k \\ g_k & h_k \end{bmatrix}, \tag{2}$$

where the subscript denotes the cycle. The matrix elements are defined by $h_k = y_1(\pi)$ and $g_k = \dot{y}_1(\pi)$ for the kth cycle, where y_1 and y_2 are the principal solutions for that cycle. Note that the matrix has only two independent elements rather than four: Since the Wronskian of the original differential equation (1) is unity, the determinant of the matrix map must be unity, and this constraint eliminates one of the independent elements. In addition, this paper specializes to the case where the periodic functions $\hat{Q}(t)$ are symmetric about the midpoint of the period, so that $y_1(\pi) = \dot{y}_2(\pi)$, which eliminates a second independent element [12]; this symmetry applies to the applications that motivated this work.

For transformation matrices $\mathcal{M}_k$ of the form (2), the eigenvalues λ_k can be used to classify the matrix types [11]. The characteristic polynomial has the form

$$\lambda_k^2 - 2h_k\lambda_k + 1 = 0. \tag{3}$$

This equation allows for three classes of eigenvalues λ_k: For $|h_k| > 1$, the eigenvalues are real and have the same sign, and the transformation matrix is hyperbolic symplectic; we denote this regime as classically unstable. When $|h_k| < 1$, the eigenvalues are complex and the matrix is elliptic; this regime is denoted as classically stable. The remaining possibility is for $|h_k| = 1$, which leads to degenerate eigenvalues equal to either $+1$ or -1; these matrices are parabolic and are stable under multiplication.

This paper studies the multiplication of infinite strings of random matrices of the form (2), i.e., the product of N such matrices in the limit $N \to \infty$. The problem of finding growth rates for infinite products of matrices with random elements was formulated over four decades ago [7, 8, 13], where existence results were given. We recall the key result here for convenience:

For a $k \times k$ matrix A with real or complex entries, let $\|A\|$ denote the Frobenius norm.

Theorem [8] *Let $X^1, X^2, X^3, \ldots$ form a metrically transitive stationary stochastic process with values in the set of $k \times k$ matrices. Suppose $\log^+ \|X^1\|$ exists, where $\log^+ t = \max(\log t, 0)$, then the limit $\lim_{N\to\infty} \|X^N X^{N-1} \cdots X^1\|$ exists.*

Determination of the growth rates are thus carried out in the limit of large N, and all probabilistic limits given here are meant almost surely.

A great deal of subsequent work has studied differential equations of the form (1) and the growth rates of the corresponding random matrices [5, 6, 10, 14, 15]. See also the paper [4]. In spite of this progress, there are relatively few examples that provide explicit expressions for the growth rates. The goal of this paper is relatively modest: It provides (what we believe to be) new analytic expressions for the growth rates of random matrices of the form (2). These expressions are derived for various regimes of parameter space, as described below.

The outline of this paper is as follows: Sect. 2 reviews the astrophysical background that led us to this topic. Section 3 considers matrix multiplication for the case where the solutions are unstable in the classical regime. Section 4 develops approximations for this regime and provides some numerical verification. Section 5 considers matrix multiplication in the regime where the solutions are classically stable. In this case, the transformation matrices $\mathcal{M}_k$ correspond to elliptical rotations and matrix multiplication is stable in the absence of fluctuations; random variations in the matrix elements render the solutions unstable. The paper concludes (in Sect. 6) with a brief summary of the results.

2 Astrophysical Background

The motivation for considering random Hill's equations arose in studies of orbit problems in astrophysics [3]. When an orbit starts in the principal plane of a triaxial, extended mass distribution (such as a dark matter halo), the motion is unstable to perturbations in the perpendicular direction. The development of the instability is described by a random Hill's equation with the form given by (1).

To illustrate this type of behavior, consider an extended mass distribution with a density profile of the form

$$\rho = \frac{\rho_0}{m} \quad \text{with } m^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}, \tag{4}$$

where ρ_0 is a density scale. This form arises in many different astrophysical contexts, including dark matter halos, galactic bulges, and young embedded star clusters. The density field is thus constant on ellipsoids, where, without loss of generality, $a > b > c > 0$. For this density profile, one can find analytic forms for both the gravitational potential and the force terms [3]. From these results, one can determine the orbital motion for a test particle moving in the potential resulting from the triaxial density distribution of (4). When the orbit begins in any of the three principal planes, the motion is generally unstable to perturbations in the perpendicular direction [1, 3]. For example, for an orbit initially confined to the x–z plane, the amplitude of the y coordinate will (usually) grow exponentially with time. In the limit of small $|y| \ll 1$, the equation of motion for the perpendicular coordinate simplifies to the form

$$\frac{d^2y}{dt^2} + \omega_y^2 y = 0 \quad \text{where } \omega_y^2 = \frac{4/b}{\sqrt{c^2x^2 + a^2z^2} + b\sqrt{x^2 + z^2}} \,. \tag{5}$$

The time evolution of the coordinates (x, z) is determined by the orbit in the original x–z plane. Since the orbital motion is nearly periodic, the $[x(t), z(t)]$ dependence of ω_y^2 represents a nearly periodic forcing term. The forcing strengths, and hence the parameters q_k appearing in Hill's equation (1), are determined by the inner turning points of the orbit (with appropriate weighting from the axis parameters $[a, b, c]$). Since the orbits are usually chaotic, the distance of closest approach, and hence the strength q_k of the forcing, varies from cycle to cycle. The outer turning points of the orbit provide a minimum value of ω_y^2, which defines the unforced oscillation frequency λ_k appearing in Hill's equation. As a result, the quantity ω_y^2 can be written in the form

$$\omega_y^2 = \lambda_k + Q_k(t), \tag{6}$$

where the index k counts the number of orbit crossings. The shapes of the functions Q_k are nearly the same, so that one can write $Q_k = q_k \hat{Q}(t)$, where $\hat{Q}(t)$ is periodic. The chaotic orbit in the original plane leads to different values of λ_k and q_k for each crossing. The equation of motion (5) for the y coordinate thus takes the form of Hill's equation (1), where the period, forcing strength, and oscillation frequency vary from cycle to cycle.

3 Matrix Multiplication for the Classically Unstable Regime

The goal of this work is to find growth rates for solutions of the differential equation (1). These growth rates are determined by multiplication of the random matrices $\mathcal{M}_k$ (from (2)) that connect solutions from cycle to cycle. These transformation matrices can also be written in the form

$$\mathcal{M}_k = h_k \mathcal{B}_k \quad \text{where } \mathcal{B}_k = \begin{bmatrix} 1 & x_k \phi_k \\ 1/x_k & 1 \end{bmatrix}, \tag{7}$$

where $x_k = h_k/g_k$ and $\phi_k = 1 - 1/h_k^2$. By virtue of our assumption on the variables (q_k, λ_k), the matrices $\mathcal{M}_k$ form a sequence of i.i.d. matrices. In this section, we consider the problem of matrix multiplication with matrices of the form (7). We specialize to the case where the solutions are unstable in the classical regime so that $|h_k| \geq 1$ and to the case where $x_k > 0$. We also assume that the h_k, x_k, and $1/x_k$ have finite means. With the matrices written in the form (7), the highly unstable regime considered in [1] can be defined as follows:

Definition Given that solutions to Hill's equation (1) are determined by transformation matrices of the form (7), the *highly unstable regime* is defined by setting $\phi_k = 1$. This specification thus defines a restricted problem.

We remark that the above regime applies when the matrix elements $|h_k| \gg 1$, which occurs for forcing strength parameters $q_k \gg 1$ [2].

The growth rates for Hill's equation (1) are determined by the growth rates for matrix multiplication of the full set of matrices $\mathcal{M}_k$. For a given matrix product, denoted here as $\mathcal{M}^{(N)}$, the *growth rate* γ is determined by

$$\gamma = \lim_{N\to\infty} \frac{1}{N} \log \|\mathcal{M}^{(N)}\|, \tag{8}$$

where the result is independent of the choice of norm $\|\cdot\|$. We note that the growth rate is called the *top* or *largest Lyapunov exponent*.

Equation (7) separates the growth rate for this problem into two parts. Let the expectation value of a sequence X_k be denoted by

$$\langle X_k \rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} X_k.$$

Then the first part γ_h of the growth rate is given by

$$\gamma_h = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \log |h_k| = \langle \log |h_k| \rangle. \tag{9}$$

We limit our discussion to distributions of the h_k for which this limit is finite. The remaining part of the growth rate is determined by matrix multiplication of the $\mathcal{B}_k$. Note that the original differential equation (1) is defined on a time interval $0 \le t \le \pi$, so that the definition of its growth rate includes a factor of π [12], whereas the growth rate for matrix multiplication (8) generally does not [8]. Ignoring these normalization issues, this paper focuses on the calculation of the growth rates for the matrices $\mathcal{M}_k$ and $\mathcal{B}_k$.

The product of N matrices of type $\mathcal{B}_k$ can be written in the form

$$\mathcal{B}^{(N)} \equiv \prod_{k=1}^{N} \mathcal{B}_k = \begin{bmatrix} \Sigma_{11} & x_1 \Sigma_{12} \\ (1/x_1)\Sigma_{21} & \Sigma_{22} \end{bmatrix}, \tag{10}$$

where the first equality defines notation and where

$$\Sigma_{11} = \sum_{j=1}^{2^{N-1}} r_j a_j, \qquad \Sigma_{12} = \sum_{j=1}^{2^{N-1}} r_j b_j,$$
$$\Sigma_{21} = \sum_{j=1}^{2^{N-1}} \frac{1}{r_j} c_j, \qquad \Sigma_{22} = \sum_{j=1}^{2^{N-1}} \frac{1}{r_j} d_j. \tag{11}$$

Here, the variables r_j are products of ratios of the form

$$r_j = \frac{x_{\mu_1} x_{\mu_2} \dots x_{\mu_n}}{x_{\nu_1} x_{\nu_2} \dots x_{\nu_n}}. \tag{12}$$

The indices are confined to the range $1 \le \mu_i, \nu_i \le N$. The additional factors a_j, b_j, c_j, d_j are products of the variables ϕ_j, and can be written in the form

$$a_j = \prod_{k=1}^{N} \phi_k^{p_k} \quad \text{where } p_k = 0 \text{ or } 1. \tag{13}$$

Result 1 For the case where $|h_k| > 1$ for all cycles, and in the limit of large N, the eigenvalue of the product matrix is given by the formula

$$\lambda = \Sigma_{11} + \Sigma_{22} + \mathcal{O}\left(h^{-2N}\right), \tag{14}$$

where each of these quantities should be labeled at the Nth iteration.

Proof The characteristic equation of the product matrix of (10) takes the form

$$\lambda^2 - \lambda(\Sigma_{11} + \Sigma_{22}) + \Sigma_{11}\Sigma_{22} - \Sigma_{12}\Sigma_{21} = 0. \tag{15}$$

The final term is the determinant of the product matrix, and this determinant is given by the product of the individual matrices, so that

$$\Sigma_{11}\Sigma_{22} - \Sigma_{12}\Sigma_{21} = \prod_{k=1}^{N}(1 - \phi_k) = \prod_{k=1}^{N} \frac{1}{h_k^2}. \tag{16}$$

Given that $|h_k| > 1$ $\forall k$, this term vanishes in the limit $N \to \infty$. As a result, the growing eigenvalue of the characteristic equation (15) simplifies to the form $\lambda = \Sigma_{11} + \Sigma_{22}$. □

Result 2 The four sums that specify the matrix elements of the product matrix are not independent. In particular, for the case where $|h_k| > 1$ and in the limit $N \to \infty$, the ratios of the matrix elements approach the form

$$\frac{\Sigma_{12}}{\Sigma_{11}} = \frac{\Sigma_{22}}{\Sigma_{21}} = \text{constant} \equiv f. \tag{17}$$

Proof As shown above, the determinant of the product matrix vanishes in the limit $N \to \infty$, so that in the limit

$$\Sigma_{11}\Sigma_{22} = \Sigma_{12}\Sigma_{21}. \tag{18}$$

The result implied by the first equality of (17) follows immediately.

Further, one can show by direct construction that if the relation of (17) holds, then the relation is preserved under matrix multiplication. Let the product matrix after N cycles have the form

$$\mathcal{B}^{(N)} = \begin{bmatrix} \Sigma_T & f x_1 \Sigma_T \\ (1/x_1)\Sigma_B & f\Sigma_B \end{bmatrix}, \tag{19}$$

where f is the constant in (17). Then the matrix takes the following from after the next cycle:

$$\mathcal{B}^{(N+1)} = \begin{bmatrix} \Sigma_T + (x/x_1)\phi\Sigma_B & x_1 f(\Sigma_T + (x/x_1)\phi\Sigma_B) \\ (1/x_1)(\Sigma_B + (x_1/x)\Sigma_T) & f(\Sigma_B + (x_1/x)\Sigma_T) \end{bmatrix}, \tag{20}$$

so that the left-right symmetry relation is conserved. □

In the above proof we have adopted notation that is used throughout this paper: The subscript '1' denotes the values of the parameters (e.g., x_1) for the first cycle in the series. Since the results of this problem can be written in terms of this starting value, these initial values play a recurring role. The subscript 'N' denotes the values of the parameters (e.g., x_N) appropriate for the Nth cycle of the series. In iteration formulae, however, we use unsubscripted variables (e.g., x) for the next $(N+1)$st cycle.

Result 3 In the highly unstable regime, the ratio of Σ_T to Σ_B has the form:

$$\frac{\Sigma_T}{\Sigma_B} = \frac{x}{x_1}. \tag{21}$$

Proof From our previous results (see (19) of [1]), the product matrix after N cycles has the form given by (19) with $f = 1$ (in the highly unstable regime). After one additional multiplication, we obtain the form given by (20) with $f = 1$. We thus find

$$\frac{{\Sigma_T}^{(N+1)}}{{\Sigma_B}^{(N+1)}} = \frac{{\Sigma_T}^{(N)} + (x/x_1){\Sigma_B}^{(N)}}{{\Sigma_B}^{(N)} + (x_1/x){\Sigma_T}^{(N)}} = \frac{x}{x_1}. \tag{22}$$

For each cycle the ratio x/x_1 has a different value, so that no limit is reached as $N \to \infty$. However, the ratio at any given finite cycle obeys (21). □

To derive an expression for the growth rate for matrix multiplication, we first define

$$S \equiv \Sigma_{11} + \Sigma_{22}. \tag{23}$$

As shown in the proof of Result 1, the eigenvalue of the product matrix approaches S, as defined above, in the limit $N \to \infty$. By construction, the iteration formula for S takes the form

$$S^{(N+1)} = S^{(N)} \left[1 + \frac{(x/x_1)\phi \Sigma_{21}{}^{(N)} + (x_1/x)\Sigma_{12}{}^{(N)}}{\Sigma_{11}{}^{(N)} + \Sigma_{22}{}^{(N)}} \right]. \tag{24}$$

Using the definition of f, Σ_T, and Σ_B, this expression can be simplified to the form

$$S^{(N+1)} = S^{(N)} \left[1 + \frac{(x/x_1)\phi \Sigma_B{}^{(N)} + (x_1/x) f \Sigma_T{}^{(N)}}{\Sigma_T{}^{(N)} + f \Sigma_B{}^{(N)}} \right]. \tag{25}$$

Result 4 In the highly unstable regime the iteration formula for the eigenvalue reduces to the form

$$S^{(N+1)} = S^{(N)} \left[1 + \frac{x_N}{x} \right]. \tag{26}$$

This result agrees with that of Theorem 2 from [1].

Proof In the highly unstable regime $\phi = 1$, $f = 1$, and (21) holds for the ratio of Σ_T/Σ_B. The iteration formula of (25) thus reduces to

$$S^{(N+1)} = S^{(N)} \left[1 + \frac{(x/x_1) + (x_N/x)}{1 + x_N/x_1} \right] = S^{(N)} \left[1 + \frac{x_N}{x} \right] \left[\frac{x_1 + x}{x_1 + x_N} \right]. \tag{27}$$

Since the starting value x_1 is fixed, the second factor in square brackets approaches unity in the limit $N \to \infty$, i.e.,

$$\lim_{N \to \infty} \prod_{k=1}^{N} \left[\frac{x_1 + x_{k+1}}{x_1 + x_k} \right] = 1. \tag{28}$$

The expression of (27) thus reduces to that of (26). □

Motivated by the result of (21) for the highly unstable regime, we write the ratio of matrix elements for the general case in the form

$$\frac{\Sigma_T{}^{(N)}}{\Sigma_B{}^{(N)}} = \frac{x_N}{x_1} \alpha_N, \tag{29}$$

so that

$$S^{(N+1)} = S^{(N)} \left[1 + \frac{(x/x_1)\phi + (x_N/x) f \alpha_N}{f + \alpha_N (x_N/x_1)} \right] \equiv \mathcal{F}_N S^{(N)}, \tag{30}$$

where the second equality defines $\mathcal{F}_N$. The parameter α_N incorporates the correction due to the matrices not being in the highly unstable regime. Note that f approaches a constant value (from Result 2) and x_1 is a constant (by definition). The iteration factor $\mathcal{F}_N$ can be rewritten in the form

$$\mathcal{F}_N = \left[1 + \frac{x^2 \phi + b \alpha_N x_N}{x(b + \alpha_N x_N)} \right] \quad \text{where } b \equiv f x_1. \tag{31}$$

Theorem 1 *The growth rate for matrix multiplication, with products of the general form defined through* (10), *is given by*

$$\gamma = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \log \left[1 + \frac{x_k^2 \phi_k + \alpha_{k-1} x_{k-1}}{x_k (1 + \alpha_{k-1} x_{k-1})} \right], \tag{32}$$

where the α_k *are determined through the iteration formula*

$$\alpha_k = \frac{x_k \phi_k + x_{k-1} \alpha_{k-1}}{x_k + x_{k-1} \alpha_{k-1}}. \tag{33}$$

Proof Note that existence of the required limit holds by the Theorem of [8]. Equations (30)–(31) show that the growth rate is given by

$$\gamma = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \log \mathcal{F}_k = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \log \left[1 + \frac{x_k^2 \phi_k + b\alpha_{k-1} x_{k-1}}{x_k (b + \alpha_{k-1} x_{k-1})} \right], \tag{34}$$

where this form is exact, provided that the α_k are properly specified. This issue is addressed below. To complete the proof, we must also show that the growth rate is independent of the value of b, so that we can set $b = 1$ in the above formula. The derivative of the growth rate with respect to the parameter b takes the form

$$\frac{d\gamma}{db} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \frac{1}{\mathcal{F}_k} \frac{d\mathcal{F}_k}{db}, \tag{35}$$

which can be evaluated to take the form

$$\frac{d\gamma}{db} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \frac{(\alpha_{k-1} x_{k-1})^2 - x_k^2 \phi_k}{(b + \alpha_{k-1} x_{k-1})[x_k (b + \alpha_{k-1} x_{k-1}) + x_k^2 \phi_k + b\alpha_{k-1} x_{k-1}]}. \tag{36}$$

This expression vanishes in the limit.

To show that the α_k are given by (33), we start with the result of matrix multiplication from (20) and use the definition of α_k from (29); these two results imply that

$$\alpha_{k+1} = \frac{x_1}{x_{k+1}} \frac{{\Sigma_T}^{(k+1)}}{{\Sigma_B}^{(k+1)}} = \frac{x_1}{x_{k+1}} \frac{{\Sigma_T}^{(k)} + (x_{k+1}/x_1)\phi_{k+1} {\Sigma_B}^{(k)}}{{\Sigma_B}^{(k)} + (x_1/x_{k+1}) {\Sigma_T}^{(k)}}. \tag{37}$$

We can then eliminate the factors of Σ_T and Σ_B by again using the definition of α_k from (29), and thus obtain

$$\alpha_{k+1} = \frac{x_1}{x_{k+1}} \frac{(x_k/x_1)\alpha_k + (x_{k+1}/x_1)\phi_{k+1}}{1 + (x_k/x_{k+1})\alpha_k} = \frac{x_k \alpha_k + x_{k+1} \phi_{k+1}}{x_{k+1} + x_k \alpha_k}. \tag{38}$$

After re-labeling the indices, we obtain (33). □

4 Approximations for the Classically Unstable Regime

For classically unstable matrices with $|h_k| > 1$, Theorem 1 provides an exact expression for the growth rate. Since the formulae are complicated, this section presents simpler but approximate expressions for the growth rates for the case where ϕ_k are small (Theorem 2) and where the differences $1 - \phi_k$ are small (Theorem 3). We also present two heuristic approximations for the growth rates for the general problem.

Theorem 2 *In the regime where the variables ϕ_k are small, $\phi_k x_k \ll 1\ \forall k$, the growth rate for the matrix $\mathcal{B}_k$ tends in the limit of large N to the form*:

$$\gamma = \log\left(1 + \left[\langle 1/x_k\rangle\langle x_k\phi_k\rangle\right]^{1/2}\right) + \mathcal{O}\left(\langle x_k\phi_k\rangle\right). \tag{39}$$

Proof We first break up the matrix into two parts so that $\mathcal{B}_k = \mathcal{I} + \mathcal{A}_k$, where $\mathcal{I}$ is the identity matrix and where

$$\mathcal{A}_k = \begin{bmatrix} 0 & x_k\phi_k \\ 1/x_k & 0 \end{bmatrix} = \begin{bmatrix} 0 & \eta_k \\ y_k & 0 \end{bmatrix}. \tag{40}$$

Note that the second equality defines $\eta_k = x_k\phi_k$ and $y_k = 1/x_k$. We first show (by induction) that repeated multiplications of the matrices $\mathcal{A}_k$ lead to products with simple forms. The products of even numbers $N = 2\ell$ of matrices $\mathcal{A}_k$ produce diagonal matrices of the form

$$\mathcal{A}^{(N)} = \mathcal{A}^{(2\ell)} = \prod_{k=1}^{N}\mathcal{A}_k = \begin{bmatrix} P_\ell^A & 0 \\ 0 & P_\ell^B \end{bmatrix}, \tag{41}$$

where the products P_ℓ are defined by

$$P_\ell^A = \prod_{i=1}^{\ell}(\eta_{2i})(y_{2i-1}) \quad \text{and} \quad P_\ell^B = \prod_{i=1}^{\ell}(\eta_{2i-1})(y_{2i}). \tag{42}$$

Similarly, the product of odd numbers $N = 2\ell + 1$ of matrices $\mathcal{A}_k$ produce off-diagonal matrices of the form

$$\mathcal{A}^{(N)} = \mathcal{A}^{(2\ell+1)} = \prod_{k=1}^{N}\mathcal{A}_k = \begin{bmatrix} 0 & Q_\ell^B\eta_1 \\ Q_\ell^A y_1 & 0 \end{bmatrix}, \tag{43}$$

where the products Q_ℓ are defined analogously to the P_ℓ. The product of N matrices $\mathcal{B}_k$ can then be written in the form

$$\mathcal{B}^{(N)} = \prod_{k=1}^{N}\mathcal{B}_k = \begin{bmatrix} \Sigma_{11} & \Sigma_{12}\eta_1 \\ \Sigma_{21}y_1 & \Sigma_{22} \end{bmatrix}. \tag{44}$$

Without loss of generality, let $N = 2\ell$ be even. Then the matrix elements are given by

$$\begin{aligned} \Sigma_{11} &= \sum_{\ell=0}^{N/2}\sum_{j=1}^{C_{2\ell}^N}\left(P_\ell^A\right)_j, \qquad \Sigma_{22} = \sum_{\ell=0}^{N/2}\sum_{j=1}^{C_{2\ell}^N}\left(P_\ell^B\right)_j, \\ \Sigma_{12} &= \sum_{\ell=0}^{N/2-1}\sum_{j=1}^{C_{2\ell+1}^N}\left(Q_\ell^B\right)_j, \qquad \Sigma_{21} = \sum_{\ell=0}^{N/2-1}\sum_{j=1}^{C_{2\ell+1}^N}\left(Q_\ell^A\right)_j, \end{aligned} \tag{45}$$

where C_ℓ^N is the binomial coefficient and where the subscripts on the P_ℓ and Q_ℓ denote different realizations of the products.

The eigenvalue Λ_N of the product matrix at the Nth iteration is given by its characteristic equation, which has the solution

$$\Lambda_N = \frac{1}{2}\left\{\Sigma_{11} + \Sigma_{22} + \left[(\Sigma_{11} - \Sigma_{22})^2 + 4\Sigma_{12}\Sigma_{21}\eta_1 y_1\right]^{1/2}\right\}. \tag{46}$$

In the limit of large N, we can make the approximation that $\Sigma_{11} \approx \Sigma_{22}$ and $\Sigma_{12} \approx \Sigma_{21}$, so that the expression for the eigenvalue takes the form

$$\Lambda_N = \Sigma_{11} + \Sigma_{12}\left[\eta_1 y_1\right]^{1/2} = \sum_{\ell=0}^{N/2}\sum_{j=1}^{C_{2\ell}^N}\left(P_\ell^A\right)_j + \sum_{\ell=0}^{N/2-1}\sum_{j=1}^{C_{2\ell+1}^N}\left(Q_\ell^B\right)_j\left[\eta_1 y_1\right]^{1/2}. \tag{47}$$

In the limit of large N, all the binomial coefficients are large except for the first and last one. We can thus rewrite the above equation in the form

$$\Lambda_N = \sum_{\ell=0}^{N/2} C_{2\ell}^N\left(\left\langle P_\ell^A\right\rangle + \varepsilon_\ell\right) + \sum_{\ell=0}^{N/2-1} C_{2\ell+1}^N\left(\left\langle Q_\ell^B\right\rangle + \varepsilon_\ell\right)\left[\eta_1 y_1\right]^{1/2}. \tag{48}$$

If the realizations of the products $(P_\ell)_j$ were independent, the error terms ε_ℓ would vanish in the limit. However, for a given N, the sums contain $C_{2\ell}^N$ terms, and $C_{2\ell}^N > N$ in general, so all of the terms in the sum cannot be independent. We then write the products $\langle P_\ell^A\rangle$ and $\langle Q_\ell^B\rangle$ in the form

$$\left\langle P_\ell^A\right\rangle + \varepsilon_\ell = \langle\eta_j\rangle^\ell\langle y_j\rangle^\ell(1+\epsilon_\ell)^\ell, \tag{49}$$

and similarly for $\langle Q_\ell^B\rangle$. This form is exact if one uses the proper expressions for the ϵ_ℓ. Using this result, the expression for the eigenvalue Λ_N becomes

$$\Lambda_N = \sum_{\ell=0}^{N/2} C_{2\ell}^N\langle\eta_j\rangle^\ell\langle y_j\rangle^\ell(1+\epsilon_\ell)^\ell + \sum_{\ell=0}^{N/2-1} C_{2\ell+1}^N\langle\eta_j\rangle^\ell\langle y_j\rangle^\ell(1+\epsilon_\ell)^\ell\left[\eta_1 y_1\right]^{1/2}, \tag{50}$$

which takes the form

$$\Lambda_N = \sum_{k=0}^{N} C_k^N\langle\eta_j\rangle^{k/2}\langle y_j\rangle^{k/2}(1+\epsilon_k)^{k/2}. \tag{51}$$

If we expand this result, we find that

$$\Lambda_N = 1 + N\langle\eta_j\rangle^{1/2}\langle y_j\rangle^{1/2}(1+\epsilon_1)^{1/2} + C_2^N\langle\eta_j\rangle\langle y_j\rangle(1+\epsilon_2) + \cdots. \tag{52}$$

Further, by performing an exact treatment of the first order expansion [2] we find that $\epsilon_1 = 0$. This finding allows us to write the product in the form

$$\Lambda_N = \left[1 + \langle\eta_j\rangle^{1/2}\langle y_j\rangle^{1/2} + \mathcal{O}(\eta_j)\right]^N. \tag{53}$$

The growth rate thus becomes

$$\gamma = \log\left[1 + \langle\eta_j\rangle^{1/2}\langle y_j\rangle^{1/2}\right] + \mathcal{O}(\eta_j). \tag{54}$$

This last expression is valid provided that $\eta_j \ll 1\ \forall j$. □

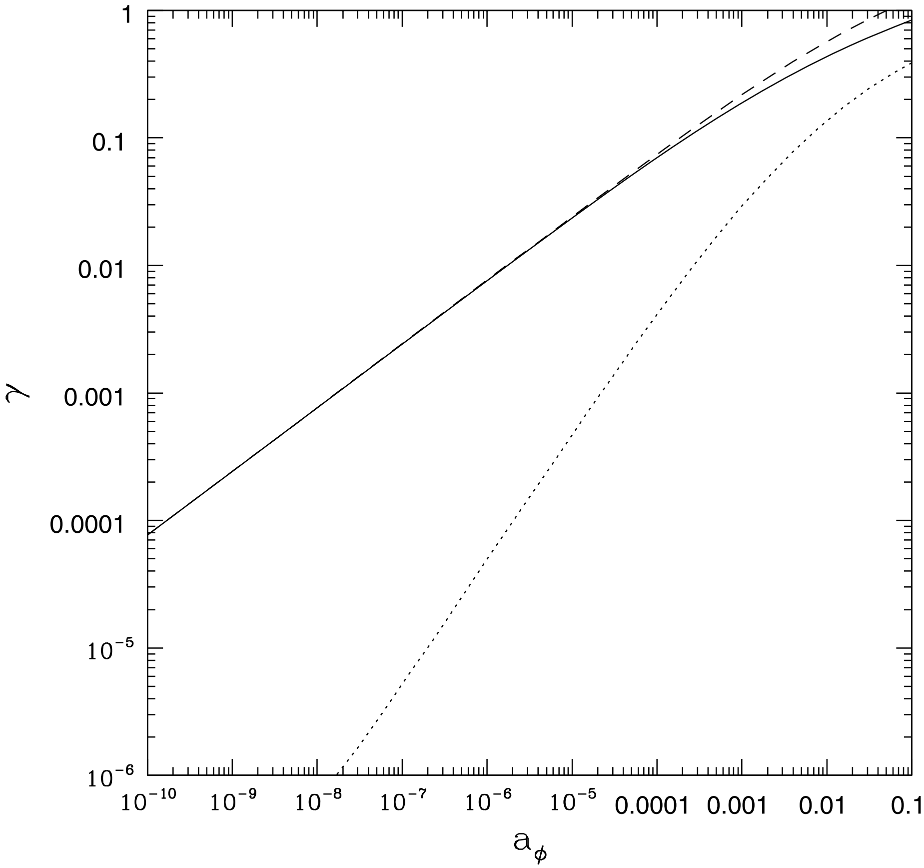

Fig. 1 Growth rates for small ϕ_k. The variables ϕ_k are determined through the relation $\phi_k = a_\phi \xi_k$, where ξ_k is uniformly distributed on [0,1]. The *solid curve* shows the growth rate γ calculated directly from matrix multiplication as a function of the amplitude a_ϕ. The *dashed curve* shows the estimate γ_2 for the growth rate from Theorem 2. The *dotted curve* shows the difference $\Delta\gamma = \gamma_2 - \gamma$. Note that $\gamma \propto \sqrt{a_\phi}$ whereas $\Delta\gamma \propto a_\phi$

Note that to consistent order, we can replace the limiting form of (39) with the equivalent, simpler function

$$\gamma \to \left[\langle 1/x_k\rangle\langle\eta_k\rangle\right]^{1/2}. \tag{55}$$

Figure 1 illustrates how well the approximation of Theorem 2 works. For the sake of definiteness, the variables x_k are log-uniformly distributed with $\log_{10} x_k \in [-2, 2]$. The ϕ_k obey the relation $\phi_k = a_\phi \xi_k$, where ξ_k is a uniformly distributed random variable over the interval $[0, 1]$. As shown by the figure, the limiting form of (39) provides an excellent description of the calculated growth rate for sufficiently small ϕ_k.

Next we consider the case where the correction factors ϕ_k are close to unity. In this case the variables $(1 - \phi_k) \ll 1$, and we can expand to leading order in $(1 - \phi_k)$. This procedure leads to the following result:

Theorem 3 *Let γ_0 be the growth rate for the highly unstable regime where $\phi_k = 1$. For small perturbations about this limiting case, the growth rate takes the form $\gamma = \gamma_0 - \delta\gamma$,* *where*

$$\delta\gamma = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \frac{(1-\phi_k)x_k^2}{(x_{k+1}+x_k)(x_k+x_{k-1})} + \mathcal{O}\left(\langle x_k^2(1-\phi_k)^2\rangle\right). \tag{56}$$

Proof We again break up the matrix into two parts,

$$\mathcal{B}_k = \mathcal{C}_k - \epsilon_k \mathcal{Z} \quad \text{with } \mathcal{Z} \equiv \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{57}$$

where here $\epsilon_k \equiv x_k(1-\phi_k)$ and $\mathcal{C}_k$ is the matrix appropriate for the highly unstable regime. Note that $\mathcal{Z}$ does not depend on the index k. Here we work to first order in the small parameter ϵ_k. After N cycles, the product matrix takes the form

$$\mathcal{B}_k^{(N)} = \prod_{k=1}^{N} \mathcal{B}_k = \mathcal{C}_k^{(N)} - \sum_{k=1}^{N} \epsilon_k \mathcal{P}_k^N + \mathcal{O}(\epsilon_k^2), \tag{58}$$

where the partial product matrices $\mathcal{P}_k^N$ are given by

$$\mathcal{P}_k^N = \left\{ \prod_{j=k+1}^{N} \mathcal{C}_j \right\} \mathcal{Z} \left\{ \prod_{j=1}^{k-1} \mathcal{C}_j \right\}. \tag{59}$$

We ignore the case where the $\mathcal{Z}$ factors appear on the ends—this effect is $\mathcal{O}(1/N)$ and vanishes in the limit. The products of the $\mathcal{C}_k$ matrices can be written in the form

$$\mathcal{C}_k^{(N)} = \Sigma_T^N \begin{bmatrix} 1 & x_1 \\ 1/x_N & x_1/x_N \end{bmatrix} \quad \text{where } \Sigma_T^N = \prod_{j=2}^{N} \left(1 + \frac{x_j}{x_{j-1}}\right), \tag{60}$$

where these results follow from previous work [1]. As a result, the matrices $\mathcal{P}_k^N$ can be evaluated:

$$\mathcal{P}_k^N = \frac{x_k \Sigma_T^N}{(x_k+x_{k+1})(x_{k-1}+x_k)} \begin{bmatrix} 1 & x_1 \\ 1/x_N & x_1/x_N \end{bmatrix} = \frac{x_k}{(x_k+x_{k+1})(x_{k-1}+x_k)} \mathcal{C}_k^{(N)}. \tag{61}$$

The product matrix $\mathcal{B}_k^{(N)}$, given by (58) to leading order, can now be written in the form

$$\mathcal{B}_k^N = \mathcal{C}_k^N \left[1 - \sum_{k=1}^{N} \frac{(1-\phi_k)x_k^2}{(x_k+x_{k+1})(x_{k-1}+x_k)} \right]. \tag{62}$$

The first factor is the product of the matrices for the highly unstable regime. Since the second factor is a function (not a matrix) its contribution to the growth rate is independent of the first factor and represents a correction to the growth rate of the form

$$\delta\gamma = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \frac{(1-\phi_k)x_k^2}{(x_k+x_{k+1})(x_{k-1}+x_k)} + \mathcal{O}(\epsilon_k^2), \tag{63}$$

where the equalities hold to leading order. This correction to the growth rate has the form given by (56). □

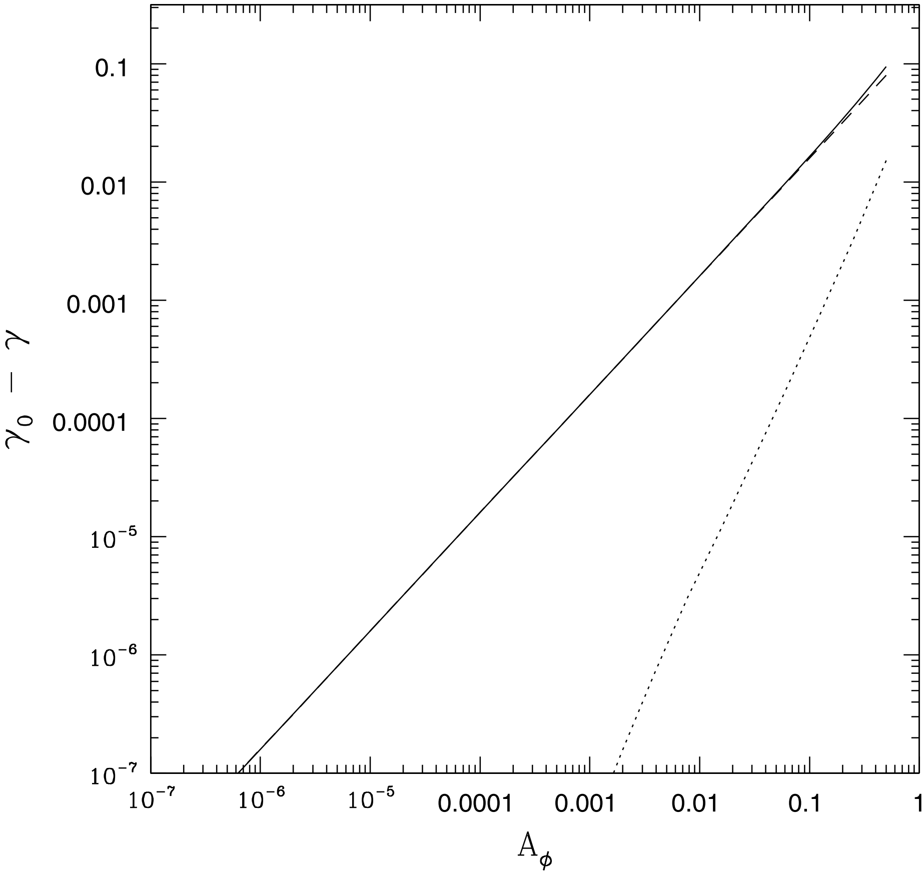

Fig. 2 Growth rates for ϕ_k near unity. The variables ϕ_k are determined through the relation $\phi_k = 1 - A_\phi \xi_k$, where ξ_k is uniformly distributed on [0,1]. The *solid curve* shows the quantity $\delta\gamma = \gamma_0 - \gamma$, where γ is the growth rate calculated from matrix multiplication and γ_0 is the growth rate for the highly unstable regime ($\phi_k = 1 \forall k$). The *dashed curve* shows the estimate $(\delta\gamma)_3 = (\gamma_0 - \gamma)_3$ for the difference in growth rate calculated from Theorem 3. The *dotted curve* shows the error $\Delta = (\delta\gamma)_3 - \delta\gamma$. Note that $\delta\gamma \propto A_\phi$ whereas the error term $\Delta \propto (A_\phi)^2$

Figure 2 shows the growth rate for small departures from the highly unstable regime. The correction factors are taken to have the form $\phi_k = 1 - A_\phi \xi_k$, where ξ_k is a uniformly distributed random variable over the interval [0, 1]. The highly unstable regime corresponds to $A_\phi \to 0$. The figure shows the growth rate calculated from direct matrix multiplication (solid curve) and the approximation from Theorem 3 (dashed curve) plotted as a function of the amplitude A_ϕ. Both curves plot the difference $\gamma_0 - \gamma$, where γ_0 is the growth rate for the highly unstable regime (where the $\phi_k = 1$).

Since the general case is quite complicated it is useful to have a good working approximation for the case where one is not in one of the two regimes ϕ_k small or near unity. Toward this end, we first show that the values of α_k have a limited range:

Result 5 The variables α_k are confined to the range $\phi_{\min} \le \alpha_k \le 1$, where $\phi_{\min}$ is the minimum value of ϕ_k.

Proof We can rewrite the iteration formula (33) for α_k in the alternate form

$$\alpha_k = \frac{\phi_k + \beta_k}{1 + \beta_k}, \tag{64}$$

where we have defined the composite random variable $\beta_k \equiv \alpha_{k-1} x_{k-1}/x_k$. In the present context, $0 \le \beta_k < \infty$, and we can show that

$$\frac{d\alpha_k}{d\beta_k} > 0 \tag{65}$$

for all values of β_k. In the limit $\beta_k \to \infty$, $\alpha_k \to 1$, whereas in the limit $\beta_k \to 0$, $\alpha_k \to \phi$. Hence $\phi \le \alpha_k \le 1$ for all cycles. But $\phi \ge \phi_{\min}$, by definition, so that $\phi_{\min} \le \alpha_k \le 1$. □

Approximation 1 As a first heuristic approximation, we replace the full iteration expression of (33) for α_k with the following simplified form

$$\alpha_{k+1} = \frac{x\phi + x_k}{x + x_k}, \tag{66}$$

i.e., we use $\alpha_k = 1$ as an approximation for the previous value [keep in mind that x is the value at the $(k+1)$th cycle]. Using (66) to evaluate α_k in the iteration formula for $\mathcal{F}_k$, we obtain a working approximation for the growth rate. Notice that α_k appears in the iteration formula for $\mathcal{F}_k$, so that we must use (66) evaluated at k rather than $k+1$. As a result, the iteration factor $\mathcal{F}_k$ involves the random variables x_k from three cycles, or, equivalently (since the x_k are i.i.d.) three separate samplings of the variables. We change notation so that x_{j1}, x_{j2}, x_{j3} denote the three independent samplings of the random variables x_k. Similarly, let ϕ_{j1}, ϕ_{j2} denote two independent samplings of the ϕ_k. The iteration formula for this approximation can then be written in the form

$$\mathcal{F}_j = 1 + \frac{x_{j1}^2 \phi_{j1}(x_{j2} + x_{j3}) + x_{j2}(x_{j2}\phi_{j2} + x_{j3})}{x_{j1}[(x_{j2} + x_{j3}) + x_{j2}(x_{j2}\phi_{j2} + x_{j3})]}. \tag{67}$$

The growth rate for matrix multiplication can then be approximated by

$$\gamma = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \log \mathcal{F}_j, \tag{68}$$

where $\mathcal{F}_j$ is given by (67). As a consistency check, for the restricted problem where the $\phi_{jn} = 1$, the iteration factor $\mathcal{F}_j$ reduces to that appropriate for the highly unstable regime (see (26)).

Approximation 2 To derive a second approximation for the growth rate, we need a better approximation for the α_k. If the values of x_k and ϕ_k were constant, then the α_k would approach a constant value given by

$$\alpha_k = \frac{1}{2}\left\{(1 - x_k/x_{k-1}) + \left[(1 - x_k/x_{k-1})^2 + 4(x_k/x_{k-1})\phi_k\right]^{1/2}\right\}. \tag{69}$$

Even though the x_k and ϕ_k are not constant, and the α_k vary, we can use (69) as an approximation to specify the values of α_k appearing in the exact formula of (32) for the growth rate.

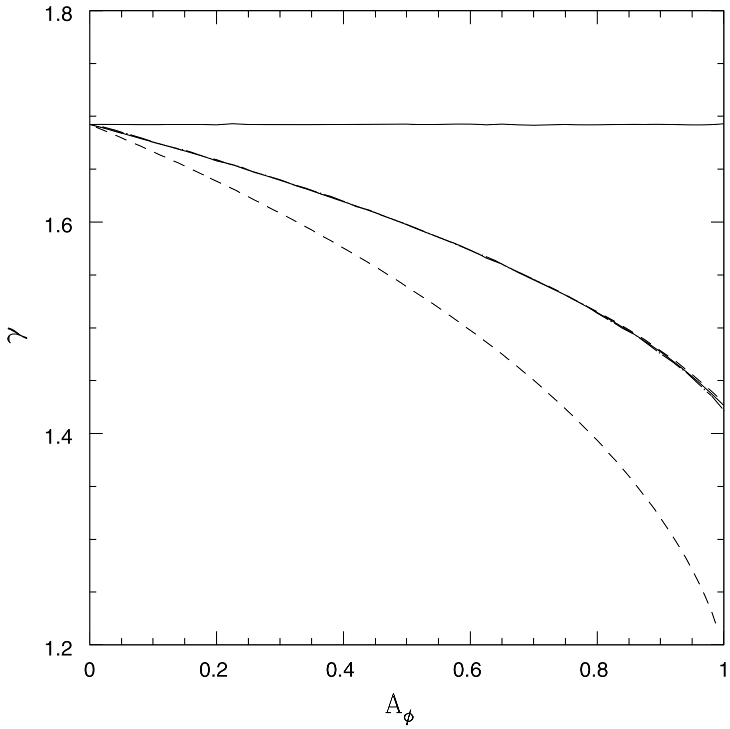

Fig. 3 Validity of approximations of (68) and (70) as a function of the deviation of ϕ_k from unity. The *upper solid line* shows the growth rate for matrix multiplication in the highly unstable regime where $\phi_k = 1$. The *lower solid curve* shows the growth rate for the case where $\phi_k = 1 - A_\phi \xi_k$, where ξ_k is a uniformly distributed random variable $0 \le \xi_k \le 1$. The *dotted curve* shows the estimate for growth rate calculated from (68) using the same sampling of the ϕ_k variables; similarly, the *dot-dashed curve* shows the approximation of (70). Notice that both of these approximations are almost identical to the actual result. The *dashed curve* shows the lower limit to the growth rate derived in [1]

After using this form to specify the α_k, and relabeling the indices, the iteration factor takes the form

$$\mathcal{F}_k = 1 + \frac{x_{k1}^2 \phi_{k1} 2x_{k3} + x_{k2}\{(x_{k3} - x_{k2}) + [(x_{k3} - x_{k2})^2 + 4x_{k2}x_{k3}\phi_{k2}]^{1/2}\}}{x_{k1}(2x_{k3} + x_{k2}\{(x_{k3} - x_{k2}) + [(x_{k3} - x_{k2})^2 + 4x_{k2}x_{k3}\phi_{k2}]^{1/2}\})}. \tag{70}$$

In the case $\phi_{jn} = 1$, the iteration factor of (70) reduces to the expression for the highly unstable regime (Result 4).

Figure 3 shows how well these two approximation schemes work. The ϕ_k variables are chosen from the expression $\phi_k = 1 - A_\phi \xi_k$, where ξ_k is a random variable uniformly sampled from the interval $0 \le \xi_k \le 1$ and where A_ϕ sets the amplitude of the departures of the ϕ_k from unity. The growth rate is shown as a function of the amplitude.

In [1], we derived a bound on the difference between the growth rate for the general case γ (considered here) and the growth rate in the highly unstable regime γ_0, i.e.,

$$\gamma_0 - \gamma \le \frac{1}{2}\langle \log \phi_k \rangle. \tag{71}$$

This bound is shown as the dashed curve in Fig. 3. The true growth rates fall comfortably between this lower bound and the growth rate for the highly unstable regime (where the latter provides an upper bound).

Thus far, this paper has focused on the regime where the transformation matrices are classically unstable. Before considering classically stable matrix multiplication in the next section, we note the following result that applies at the transition between the two regimes:

Result 6 Consider the matrix transformation that maps the principal solutions from one cycle to the next. When the matrix elements $g_k = \dot{y}_1(\pi)$ vanish, then the remaining matrix elements are $h_k = y_1(\pi) = \pm 1$. The transformation matrix $\mathcal{M}_{g0}$ for this case is stable under multiplication.

(The proof is a simple explicit computation.)

5 Elliptical Rotations and the Classically Stable Regime

When the principal solutions h_k appearing in the discrete map of (2) are less than unity, matrix multiplication is stable for the case of constant parameters. In the case of interest, however, the parameters in Hill's equation (1) and the matrices (2) vary from cycle to cycle. This section considers the case where the $|h_k| \le 1$, but vary from cycle to cycle, and show that instability results. In this regime, the discrete map takes the form of an elliptical rotation matrix [11] as described below. We thus find the growth rates for elliptical rotation matrices for the case where the matrix elements vary from cycle to cycle.

Definition An *elliptical rotation matrix* is defined to be

$$\mathcal{E}(\theta; L) \equiv \begin{bmatrix} \cos\theta & -L\sin\theta \\ (1/L)\sin\theta & \cos\theta \end{bmatrix}. \tag{72}$$

These matrices have the following properties:
The product of elliptical rotation matrices with the same value of L produces another elliptical rotation matrix, also with the same L,

$$\mathcal{E}(\theta_1; L)\mathcal{E}(\theta_2; L) = \mathcal{E}\left([\theta_1 + \theta_2]; L\right). \tag{73}$$

As a result, the elliptical rotation matrices form a group.
For fixed L, matrix multiplication is stable. Specifically, the eigenvalues of the product of N matrices (with fixed L) have the form

$$\lambda = \exp\left[\pm i \sum_{j=1}^{N} \theta_j\right] = \exp\left[\pm i\theta_N\right], \tag{74}$$

where θ_N is the angle corresponding to the group element produced after N matrix multiplications.

Result 7 When an individual cycle of Hill's equation is stable, specifically when $|h_k| \le 1$, the full transformation matrix $\mathcal{M}_k$ takes the form of an elliptical rotation.

Proof Since $|h_k| \le 1$, we can define an angle θ_k such that $h_k = \cos\theta_k$. The full matrix $\mathcal{M}_k$ given by (7) then takes the form

$$\mathcal{M}_k = \begin{bmatrix} \cos\theta_k & -(\sin^2\theta_k)/g_k \\ g_k & \cos\theta_k \end{bmatrix} = \begin{bmatrix} \cos\theta_k & -L_k \sin\theta_k \\ (1/L_k)\sin\theta_k & \cos\theta_k \end{bmatrix} = \mathcal{E}_k(\theta_k; L_k), \tag{75}$$

where we have defined $L_k = (\sin\theta_k)/g_k$. As before, we can factor out the $\cos\theta_k = h_k$ and write the matrix in the form

$$\mathcal{M}_k = \cos\theta_k \begin{bmatrix} 1 & x_k\phi_k \\ 1/x_k & 1 \end{bmatrix} = \cos\theta_k \mathcal{B}_k, \tag{76}$$

where

$$x_k = L_k/\tan\theta_k \quad \text{and} \quad \phi_k = -\tan^2\theta_k. \tag{77}$$

Equation (77) thus specifies the transformation between the random variables (x_k, ϕ_k) appearing in the original transformation matrix and the random variables (θ_k, L_k) in the corresponding elliptical rotation matrix. Note that the values of ϕ_k are strictly negative in this formulation. Otherwise, the matrix $\mathcal{B}_k$ has the same form as in (7). □

If we let γ_B be the growth rate for matrix $\mathcal{B}_k$, then the growth rate γ_M for the full matrix $\mathcal{M}_k$ takes the form

$$\gamma_M = \gamma_B + \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \log[\cos\theta_k]. \tag{78}$$

The exact growth rate for the matrix $\mathcal{B}_k$ (see (76)) is given by Theorem 1. In particular, (32) and (33) remain valid for negative values of the ϕ_k and can be used to calculate the growth rate.

Result 8 For an elliptical rotation matrix with constant angle θ and random L_k, the growth rate for matrix multiplication vanishes in the two limits $h = \cos\theta \to 0$ and $h = \cos\theta \to 1$.

Proof In the limit $h \to 1$ we have $\sin\theta = 0$, and the elliptical rotation matrix becomes the identity matrix. As a result, the growth rate vanishes.
In the other case where $h \to 0$, $\sin\theta = 1$, and the matrix takes the form

$$\mathcal{E}_k \to \mathcal{E}_{0k} = \begin{bmatrix} 0 & -L_k \\ 1/L_k & 0 \end{bmatrix}. \tag{79}$$

In this case, for even numbers of matrix multiplications, say $N = 2n$, the product matrix takes the form

$$\mathcal{E}_{0k}^{(N)} = \prod_{k=1}^{N} \mathcal{E}_{0k} = (-1)^n \begin{bmatrix} P_n^A & 0 \\ 0 & P_n^B \end{bmatrix}, \tag{80}$$

where the matrix elements are given by the products

$$P_n^A = \prod_{k=1}^{n} \frac{L_{2k}}{L_{2k-1}} \quad \text{and} \quad P_n^B = \prod_{k=1}^{n} \frac{L_{2k-1}}{L_{2k}}. \tag{81}$$

The eigenvalues of the product matrix are given by $\lambda = P_n^A$ and $\lambda = P_n^B$. For odd $N = 2n + 1$, the eigenvalue $|\lambda| = (P_n^A P_n^B)^{1/2}$. In either case, in the limit of large N, the growth rate for matrix multiplication takes the form

$$\gamma = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \log\left[\frac{L_{2k}}{L_{2k-1}}\right] = \langle \log L_{2k}\rangle - \langle \log L_{2k-1}\rangle = 0. \tag{82}$$

The final equality holds because the L_k are independent. □

Elliptical rotation matrices are unstable under multiplication when their parameters vary from cycle to cycle:

Theorem 4 *Consider an elliptical rotation matrix with variable angle θ_k and symmetric fluctuations of the L_k parameter about its mean value L_0. The variations are thus written in the form $L_k = L_0(1+\eta_k)$, where the odd moments $\langle \eta_k^{2n+1}\rangle = 0$ for all integers n. For small fluctuations $|\eta_k| < 1$, the growth γ rate for matrix multiplication takes the form*

$$\gamma = \frac{1}{2} \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \log\left[\cos^2\theta_k + \sin^2\theta_k \left\langle \frac{1}{1+\eta_j}\right\rangle\right] + \mathcal{O}\left(\eta_k^4\right). \tag{83}$$

Proof We first break up the matrix into two parts so that

$$\mathcal{E}_k = \mathcal{I}\cos\theta_k + \sin\theta_k \mathcal{Z}_k, \tag{84}$$

where $\mathcal{I}$ is the identity matrix and where

$$\mathcal{Z}_k = \begin{bmatrix} 0 & -L_k \\ 1/L_k & 0 \end{bmatrix}. \tag{85}$$

The product of N matrices $\mathcal{E}_k$ becomes

$$\mathcal{E}^{(N)} = \prod_{k=1}^{N} \mathcal{E}_k = \sum_{\ell=0}^{N} \sum_{k=1}^{C_\ell^N} \left(\prod_{i=1}^{N-\ell} \cos\theta_i\right)_k \left(\prod_{j=1}^{\ell} \mathcal{Z}_j \sin\theta_j\right)_k, \tag{86}$$

where the subscripts on the products denote different realizations. The products of even numbers $\ell = 2n$ of matrices $\mathcal{Z}_k$ produce diagonal matrices of the form

$$\mathcal{Z}^{(\ell)} = \mathcal{Z}^{(2n)} = \prod_{k=1}^{n} \mathcal{Z}_{2k}\mathcal{Z}_{2k-1} = (-1)^n \begin{bmatrix} P_n^A & 0 \\ 0 & P_n^B \end{bmatrix}, \tag{87}$$

where the matrix elements P_n^A and P_n^B are given by (81). Similarly, the product of odd numbers $\ell = 2n+1$ of matrices $\mathcal{Z}_k$ produce off-diagonal matrices of the form

$$\mathcal{Z}^{(\ell)} = \mathcal{Z}^{(2n+1)} = \left\{\prod_{k=1}^{n} \mathcal{Z}_{2k+1}\mathcal{Z}_{2k}\right\} \mathcal{Z}_1 = (-1)^n \begin{bmatrix} 0 & -P_n^A L_1 \\ P_n^B/L_1 & 0 \end{bmatrix}, \tag{88}$$

where the P_n are defined previously. Next we write the expectation values of these products in the form

$$\langle P_n \rangle = \left\langle \prod_{j=1}^{n} \frac{L_{2j}}{L_{2j-1}} \right\rangle = \left\langle \prod_{j=1}^{n} \frac{1+\eta_{2j}}{1+\eta_{2j-1}} \right\rangle = \left\langle \frac{1}{1+\eta_j} \right\rangle^n \equiv \mathcal{R}^n. \tag{89}$$

This expression holds because the odd powers of the η_j vanish in the mean, and the samples of the different η's are independent.

The eigenvalue Λ_N of the product matrix at the Nth iteration can be written in terms of its matrix elements, i.e.,

$$\Lambda_N = \sigma_{11} + \sigma_{22}. \tag{90}$$

Without loss of generality, let $N = 2K$ be even. The matrix elements $\sigma_{11} = \sigma_{22} = \sigma$ are given by

$$\sigma = \sum_{m=0}^{K} \sum_{k=1}^{C_{2m}^{2K}} \left(\prod_{i=1}^{2K-2m} \cos\theta_i \right)_k \left(\prod_{i=1}^{2m} \sin\theta_i \right)_k (-1)^m \mathcal{R}^m, \tag{91}$$

where C_{2m}^{2K} is the binomial coefficient and where we have used (89). This expression for σ contains the even terms of a binomial expansion. We can thus write the eigenvalue in the form

$$\Lambda_N = \prod_{k=1}^{N} \left[\cos\theta_k + i\sin\theta_k \mathcal{R}^{1/2}\right]_k + \prod_{k=1}^{N} \left[\cos\theta_k - i\sin\theta_k \mathcal{R}^{1/2}\right]_k. \tag{92}$$

Next we define

$$A_k \equiv \left[\cos^2\theta_k + \sin^2\theta_k \mathcal{R}\right]^{1/2} \quad \text{and} \quad \tan\alpha_k \equiv \mathcal{R}^{-1/2}\tan\theta_k. \tag{93}$$

The eigenvalue takes the form

$$\Lambda_N = 2\left(\prod_{k=1}^{N} A_k\right) \cos\left(\sum_{k=1}^{N} \alpha_k\right), \tag{94}$$

and the corresponding growth rate becomes

$$\gamma = \frac{1}{2} \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \log\left[\cos^2\theta_k + \sin^2\theta_k \mathcal{R}\right]. \tag{95}$$

Using the definition of $\mathcal{R}$, we obtain the result of Theorem 4. The order of the error term follows by comparing (95) with the leading order expansion [2]. □

In the regime of small $\eta_k \ll 1$, the expression for the growth rate reduces to the form

$$\gamma = \frac{1}{2} \langle \sin^2\theta_k \rangle \langle \eta_k^2 \rangle. \tag{96}$$

This section shows that instability does not require a finite threshold for the amplitude of the fluctuations in L_k. Nonzero amplitude leads to instability with growth rate $\gamma \propto \langle \eta_k^2 \rangle$. Variations in the original parameters (λ_k, q_k) of Hill's equation lead to fluctuations in the principal solutions (h_k, g_k); fluctuations in the (h_k, g_k) lead to variations in the L_k and hence growth. As a result, Hill's equation with random forcing terms is generically unstable. One notable exception occurs when the $h_k = 0$ or $h_k = 1$ (Result 8).

6 Conclusion

This paper provides expressions for the growth rates for the random 2×2 matrices that result from solutions to the random Hill's equation (1). Theorem 1 gives an exact expression for the growth rate. Theorems 2 and 3 provide approximate growth rates for the regimes where the variables ϕ_k are small, and close to unity, respectively. Additional approximations for are given in Sect. 4. When Hill's equation is classically stable, the discrete map that governs the solutions has the form of an elliptical rotation matrix (72). With fixed elements, such matrices are stable under multiplication; variations in the L_k parameter lead to instability. For small symmetric fluctuations of the length parameter L_k, the growth rate is given by Theorem 4.

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